

Multi-bump solutions for a Kirchhoff problem type

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Abstract

In this paper, we are going to study the existence of solution for the following Kirchhoff problem

$$\begin{cases} M\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} \lambda a(x) + 1) u^2 dx\right) \left(-\Delta u + (\lambda a(x) + 1)u\right) = f(u) & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3). \end{cases}$$

Assuming that the nonnegative function $a(x)$ has a potential well with $\text{int}(a^{-1}(\{0\}))$ consisting of k disjoint components $\Omega_1, \Omega_2, \dots, \Omega_k$ and the nonlinearity $f(t)$ has a subcritical growth, we are able to establish the existence of positive multi-bump solutions by variational methods.

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1 Introduction

In the present paper, we are interested in showing the existence of multi-bump solution for the following class of Kirchhoff problem

$$\begin{cases} M\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} (\lambda a(x) + 1) u^2 dx\right) \left(-\Delta u + (\lambda a(x) + 1)u\right) = f(u) & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \quad (P)_\lambda$$

where $\lambda > 0$ is a positive parameter and M, a and f are functions verifying some conditions, which will be fixed below.

The function $M : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to $C^1(\mathbb{R}, \mathbb{R})$ and satisfies the following conditions:

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(M₁) The function M is increasing and $0 < M(0) =: m_0$.

(M₂) The function $t \mapsto \frac{M(t)}{t}$ is decreasing.

A typical example of function verifying the assumptions (M₁) – (M₂) is given by

$$M(t) = m_0 + bt, \quad \text{where } m_0 > 0 \quad \text{and } b > 0.$$

This is the example that was considered in [21]. More generally, each function of the form

$$M(t) = m_0 + bt + \sum_{i=1}^k b_i t^{\gamma_i}$$

with $b_i \geq 0$ and $\gamma_i \in (0, 1)$ for all $i \in \{1, 2, \dots, k\}$ verifies the hypotheses (M₁) – (M₂). An another example is $M(t) = m_0 + \ln(1 + t)$.

Related to function $a(x)$, we assume the following conditions:

(a₀) $a(x) \geq 0, \quad \forall x \in \mathbb{R}^N$.

(a₁) The set $\text{int}(a^{-1}(\{0\}))$ is nonempty and there are disjoint open components $\Omega_1, \Omega_2, \dots, \Omega_k$ such that

$$\text{int}(a^{-1}(\{0\})) = \cup_{j=1}^k \Omega_j \tag{1.1}$$

and

$$\text{dist}(\Omega_i, \Omega_j) > 0 \quad \text{for } i \neq j, \quad i, j = 1, 2, \dots, k. \tag{1.2}$$

Finally, the function f is a continuous function satisfying:

$$(f_1) \quad \lim_{s \rightarrow 0} \frac{f(s)}{s} = 0,$$

$$(f_2) \quad \lim_{|s| \rightarrow +\infty} \frac{f(s)}{s^5} = 0,$$

(f₃) There exists $\theta > 4$ such that

$$0 < \theta F(s) \leq s f(s) \quad \forall s \in \mathbb{R} \setminus \{0\}.$$

(f₄) $\frac{f(s)}{s^3}$ is increasing in $s > 0$ and decreasing in $s < 0$.

Related to problem $(P)_\lambda$, we have the problem

$$(*) \quad \begin{cases} -M \left(\int_{\Omega} |\nabla u|^2 dx \right) \Delta u = f(u) \text{ in } \Omega, \\ u \in H_0^1(\Omega) \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain. This type of problem is called *Kirchhoff problem*, because of the presence of the term $M \left(\int_{\Omega} |\nabla u|^2 dx \right)$. Indeed, this operator appears in

the Kirchhoff equation [21], which arises in nonlinear vibrations, namely

$$\begin{cases} u_{tt} - M\left(\int_{\Omega} |\nabla u|^2 dx\right) \Delta u = g(x, u) & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ u(x, 0) = u_0(x) \quad , \quad u_t(x, 0) = u_1(x). \end{cases}$$

The reader may consult [2], [3], [15], [27] and the references therein, for more physical motivation on Kirchhoff problem.

We would like point out that in the last years many authors have studied this type of problem in bounded or unbounded domains, see for example, [4], [6], [8], [9], [10], [11], [14], [15], [16], [17], [20], [23], [24], [25], [26], [31], [32], [34], [35], [36] and reference therein. For solutions that change sign (nodal solution) we would like to cite [18], [28], [29], [33] and [38].

The motivation to study the problem $(P)_{\lambda}$ comes from of a paper due to Ding and Tanaka [13], which has studied $(P)_{\lambda}$ assuming $M(t) = 1$ and $f(t) = |t|^{q-1}t$. In that interesting paper, the authors considered the existence of positive multi-bump solution for the problem

$$\begin{cases} -\Delta u + (\lambda a(x) + Z(x))u = u^q & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \quad (1.3)$$

$q \in (1, \frac{N+2}{N-2})$ if $N \geq 3$; $q \in (1, \infty)$ if $N = 1, 2$. The authors showed that the above problem has at least $2^k - 1$ solutions u_{λ} for large values of λ . More precisely, for each non-empty subset Υ of $\{1, \dots, k\}$, it was proved that, for any sequence $\lambda_n \rightarrow \infty$ we can extract a subsequence (λ_{n_i}) such that $(u_{\lambda_{n_i}})$ converges strongly in $H^1(\mathbb{R}^N)$ to a function u , which satisfies $u = 0$ outside $\Omega_{\Upsilon} = \bigcup_{j \in \Upsilon} \Omega_j$ and $u|_{\Omega_j}$, $j \in \Upsilon$, is a least energy solution for

$$\begin{cases} -\Delta u + Z(x)u = u^q, & \text{in } \Omega_j, \\ u \in H_0^1(\Omega_j), u > 0, & \text{in } \Omega_j. \end{cases} \quad (1.4)$$

Involving the Kirchhoff problem with potential wells, there are not so many existing papers. As far as we know, the only paper that considered the existence of solutions for $(P)_{\lambda}$ is due to Liang and Shi [22]. Unfortunately, we believe that the Section V of the above paper has a mistake, which commits the proof of the their main result, to be more precisely, we have observed that the numbers c_j and $c_{\lambda,j}$ considered in that work are not a good choice for this class of problem, and also, the proof of Lemma 5.3 is not correct, because the authors have forgotten that the Kirchhoff problem has a nonlocal term involving the function M , which is very sensitive for some estimates. Motivated by [13] and [22], we intend in the present paper to show how we can work with this nonlocal term to get a positive multi-bump solution for $(P)_{\lambda}$. Here, we will adapt an idea used by Alves and Yang [5] to show the existence of multi-bump solution for the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + (\lambda a(x) + 1)u + \phi u = f(u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases}$$

Our main result is the following

Theorem 1.1. *Assume that $(M_1), (M_2), (a_0), (a_1)$ and $(f_1) - (f_4)$ hold. Then, there exist $\lambda_0 > 0$ with the following property: for any non-empty subset Υ of $\{1, 2, \dots, k\}$ and $\lambda \geq \lambda_0$, problem (P_λ) has a positive solution u_λ . Moreover, if we fix the subset Υ , then for any sequence $\lambda_n \rightarrow \infty$ we can extract a subsequence (λ_{n_i}) such that $(u_{\lambda_{n_i}})$ converges strongly in $H^1(\mathbb{R}^3)$ to a function u , which satisfies $u = 0$ outside $\Omega_\Upsilon = \cup_{j \in \Upsilon} \Omega_j$, and $u|_{\Omega_\Upsilon}$ is a least energy solution for the nonlocal problem*

$$\begin{cases} M \left(\int_{\Omega_\Upsilon} |\nabla u|^2 dx + \int_{\Omega_\Upsilon} u^2 dx \right) (-\Delta u + u) = f(u) & \text{in } \Omega_\Upsilon, \\ u(x) > 0 \quad \forall x \in \Omega_j \text{ and } \forall j \in \Upsilon, \\ u \in H_0^1(\Omega_\Upsilon). \end{cases} \quad (P)_{\infty, \Upsilon}$$

The paper is organized as follows. In the next section, we prove some technical lemmas and the existence of least energy solution for $(P)_{\infty, \Upsilon}$. In Section 3, we study an auxiliary problem. A compactness result for the energy functional associated with the auxiliary problem is showed in Section 4. Some estimates involving the solutions of auxiliary problem are showed in Section 5, and in Section 6, we build a special minimax value for the functional energy associated to the auxiliary problem.

2 The problem $(P)_{\infty, \Upsilon}$

In the sequel, let us denote by \widehat{M} and F the following functions

$$\widehat{M}(t) = \int_0^t M(s) ds \quad \text{and} \quad F(t) = \int_0^t f(s) ds.$$

In the proof of Theorem 1.1, we need to study the existence of least energy solution for problem $(P)_{\infty, \Upsilon}$. The main idea is to prove that the energy functional J associated with nonlocal problem $(P)_{\infty, \Upsilon}$ given by

$$J(u) = \frac{1}{2} \widehat{M} \left(\int_{\Omega_\Upsilon} (|\nabla u|^2 + |u|^2) dx \right) - \int_{\Omega_\Upsilon} F(u) dx,$$

assumes a minimum value on the set

$$\mathcal{M}_\Upsilon = \{u \in \mathcal{N}_\Upsilon : J'(u)u_j = 0 \text{ and } u_j \neq 0 \quad \forall j \in \Upsilon\}$$

where $u_j = u|_{\Omega_j}$ and \mathcal{N}_Υ is the corresponding Nehari manifold defined by

$$\mathcal{N}_\Upsilon = \{u \in H_0^1(\Omega_\Upsilon) \setminus \{0\} : J'(u)u = 0\}.$$

More precisely, we will prove that there is $w \in \mathcal{M}_\Upsilon$ such that

$$J(w) = \inf_{u \in \mathcal{M}_\Upsilon} J(u).$$

After, we use the implicit function theorem to prove that w is a critical point of J , and so, w is a least energy solution for $(P)_{\infty, \Upsilon}$. The main feature of the least energy solution

w is that $w(x) > 0 \ \forall x \in \Omega_j$ and $\forall j \in \Upsilon$, which will be used to describe the existence of multi-bump solutions.

Since we intend to look for positive solutions, throughout this paper we assume that

$$f(s) = 0, \quad s \leq 0.$$

Moreover, notice that by (f_1) and (f_2) , given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$f(t)t \leq \epsilon|t|^2 + C_\epsilon|t|^6. \quad (2.1)$$

In what follows, to show in details the idea of the existence of least energy solution for $(P)_{\infty, \Upsilon}$, we will consider $\Upsilon = \{1, 2\}$. Moreover, we will denote by Ω , \mathcal{N} and \mathcal{M} the sets Ω_Υ , \mathcal{N}_Υ and \mathcal{M}_Υ respectively. Thereby,

$$\Omega = \Omega_1 \cup \Omega_2,$$

$$\mathcal{N} = \{u \in H_0^1(\Omega) \setminus \{0\} : J'(u)u = 0\}$$

and

$$\mathcal{M} = \{u \in \mathcal{N} : J'(u)u_1 = J'(u)u_2 = 0 \text{ and } u_1, u_2 \neq 0\},$$

with $u_j = u|_{\Omega_j}$, $j = 1, 2$.

2.1 Technical lemmas

Hereafter, let us denote by $\|\cdot\|$, $\|\cdot\|_1$ and $\|\cdot\|_2$ the norms in $H_0^1(\Omega)$, $H_0^1(\Omega_1)$ and $H_0^1(\Omega_2)$ given by

$$\|u\| = \left(\int_{\Omega} (|\nabla u|^2 + |u|^2) dx \right)^{\frac{1}{2}},$$

$$\|u\|_1 = \left(\int_{\Omega_1} (|\nabla u|^2 + |u|^2) dx \right)^{\frac{1}{2}}$$

and

$$\|u\|_2 = \left(\int_{\Omega_2} (|\nabla u|^2 + |u|^2) dx \right)^{\frac{1}{2}}$$

respectively.

In order to show that the set \mathcal{M} is not empty, we need of the following Lemma.

Lemma 2.1. *Let $v \in H_0^1(\Omega)$ with $v_j \neq 0$ for $j = 1, 2$. Then, there are $t, s > 0$ such that $J'(tv_1 + sv_2)v_1 = 0$ and $J'(tv_1 + sv_2)v_2 = 0$.*

Proof. Let $V : (0, +\infty) \times (0, +\infty) \rightarrow \mathbb{R}^2$ be a continuous function given by

$$V(t, s) = (J'(tv_1 + sv_2)(tv_1), J'(tv_1 + sv_2)(sv_2)).$$

Note that

$$J'(tv_1 + sv_2)(tv_1) = t^2 M(t^2 \|v_1\|_1^2 + s^2 \|v_2\|_2^2) \|v_1\|_1^2 - \int_{\Omega_1} f(tv_1) tv_1 dx. \quad (2.2)$$

Using (M_1) , (2.1) and Sobolev's embedding in (2.2), we have

$$J'(tv_1 + sv_2)(tv_1) \geq (m_0 - \epsilon C)t^2\|v_1\|_1^2 - t^q C_\epsilon C\|v_1\|_1^6,$$

for some $C > 0$. Thus, there exists $r > 0$ sufficiently small such that

$$J'(rv_1 + sv_2)(rv_1) > 0, \quad \text{for all } s > 0.$$

The same idea yields

$$J'(tv_1 + rv_2)(rv_2) > 0, \quad \text{for all } t > 0.$$

On the other hand, by (M_2) , there exists $K_1 > 0$ such that

$$M(t) \leq M(1)t + K_1, \quad \text{for all } t \geq 0 \quad (2.3)$$

and by (f_3) , there are $K_2, K_3 > 0$ such that

$$F(t) \geq K_2 t^\theta - K_3. \quad (2.4)$$

Using (2.3), (2.4) and (f_3) in (2.2), we derive that

$$\begin{aligned} J'(tv_1 + sv_2)(tv_1) &\leq t^4 M(1)\|v_1\|_1^4 + t^2 s^2 M(1)\|v_1\|_1^2\|v_2\|_2^2 + K_1 t^2\|v_1\|_1^2 \\ &\quad - \frac{t^\theta}{\theta} K_2 \int_{\Omega} |v_1|_1^\theta dx + K_3 |\Omega_1|, \end{aligned}$$

where $|\Omega_1|$ denotes the Lebesgue measure of Ω_1 . Thus, since $\theta > 4$, for $R > 0$ sufficiently large, we get

$$J'(Rv_1 + sv_2)(Rv_1) < 0, \quad \text{for all } s \leq R.$$

Arguing of the same way, we also have

$$J'(tv_1 + Rv_2)(Rv_2) < 0, \quad \text{for all } t \leq R.$$

In particular,

$$J'(rv_1 + sv_2)(rv_1) > 0 \quad \text{and} \quad J'(tv_1 + rv_2)(rv_2) > 0, \quad \text{for all } t, s \in [r, R]$$

and

$$J'(Rv_1 + sv_2)(Rv_1) < 0 \quad \text{and} \quad J'(tv_1 + Rv_2)(Rv_2) < 0, \quad \text{for all } t, s \in [r, R].$$

Now the lemma follows applying Miranda's Theorem [30]. \square

As an immediate consequence of the last lemma, we have the following corollary

Corollary 2.2. *The set \mathcal{M} is not empty.*

Next, we will show some technical lemmas.

Lemma 2.3. *There exists $\rho > 0$ such that*

- (i) $J(u) \geq \frac{(\theta-4)}{4\theta} m_0 \|u\|^2$ and $\|u\| \geq \rho, \forall u \in \mathcal{N}$;
(ii) $\|w_j\|_j \geq \rho, \forall w \in \mathcal{M}$ and $j = 1, 2$, where $w_j = w|_{\Omega_j}, j = 1, 2$.

Proof. From definition of \widehat{M} and (M_2) ,

$$\widehat{M}(t) \geq \frac{1}{2} M(t)t, \text{ for all } t \geq 0. \quad (2.5)$$

Now, a simple computation together with (2.5) gives

$$\frac{1}{2} \widehat{M}(t) - \frac{1}{\theta} M(t)t \geq \frac{(\theta-4)}{4\theta} m_0 t, \text{ for all } t \geq 0. \quad (2.6)$$

Thus, by (f_3) and (2.6),

$$J(u) = J(u) - \frac{1}{\theta} J'(u)u \geq \frac{(\theta-4)}{4\theta} m_0 \|u\|^2, \text{ for all } u \in \mathcal{N}.$$

Gathering definition of \mathcal{N} , (M_1) , (2.1) and Sobolev's embedding, it follows that

$$0 < \rho := \left[\left(m_0 - \frac{\epsilon C_2}{C_1} \right) \frac{1}{C_\epsilon} \right]^{1/(q-2)} \leq \|u\|,$$

for all $u \in \mathcal{N}$ and for some $C_1, C_1 > 0$.

From (M_1) ,

$$M(\|w_j\|_j^2) \leq M(\|w\|^2), \quad \forall w \in \mathcal{M}.$$

Thus,

$$J'(w_j)w_j \leq 0, \text{ for all } w \in \mathcal{M}, \quad (2.7)$$

implying that

$$0 < \rho \leq \|w_j\|_j.$$

□

Lemma 2.4. *If (w_n) is a bounded sequence in \mathcal{M} and $q \in (2, 6)$, we have*

$$\liminf_n \int_{\Omega_j} |w_{n,j}|^p dx > 0 \quad j = 1, 2.$$

where $w_{n,j} = w_n|_{\Omega_j}$ for $j = 1, 2$.

Proof. Notice that by (f_1) and (f_2) , given $\epsilon > 0$, there exist $C > 0$ and $q \in (2, 6)$ such that

$$f(t)t \leq \epsilon |t|^2 + C |t|^q + \epsilon |t|^6. \quad (2.8)$$

Therefore,

$$0 < m_0 \rho^2 \leq M(\|w_{n,j}\|_j^2) \|w_{n,j}\|_j^2 \leq \epsilon \int_{\Omega_j} |w_{n,j}|^2 dx + C \int_{\Omega} |w_{n,j}|^q dx + \epsilon \int_{\Omega} |w_{n,j}|^6 dx.$$

Since (w_n) is bounded, there is $\tilde{C} > 0$ such that

$$0 < m_0 \rho^2 \leq \epsilon \tilde{C} + C \int_{\Omega} |w_{n,j}|^q dx.$$

Now, the result follows choosing ϵ small enough. □

2.2 Existence of least energy solution for $(P)_{\infty, \Upsilon}$

At this point, some useful remarks follow. First of all, let us observe that, from (M_2) ,

$$M'(t)t \leq M(t), \quad \text{for all } t \geq 0, \quad (2.9)$$

from where it follows that

$$t \mapsto \frac{1}{2}\widehat{M}(t) - \frac{1}{4}M(t)t \text{ is increasing.} \quad (2.10)$$

Now, by (f_4) ,

$$f'(t)t \geq 3f(t), \quad \text{for all } |t| \geq 0, \quad (2.11)$$

implying that

$$t \mapsto \frac{1}{4}f(t)t - F(t) \text{ is increasing, for all } |t| > 0. \quad (2.12)$$

In this subsection, our main goal is to prove the following result:

Theorem 2.5. *Assume that $(f_1) - (f_4)$ hold. Then problem $(P)_{\infty, \Upsilon}$ possesses a positive least energy solution on the set \mathcal{M} .*

Proof. We will prove the existence of $w \in \mathcal{M}$ in which the infimum of J is attained on \mathcal{M} . After, using the implicit function theorem, we show that w is a critical point of J , from where it follows that w is a least energy solution of $(P)_{\infty, \Upsilon}$.

First of all, by Lemma 2.3, there exists $c_0 \in \mathbb{R}$ such that

$$0 < c_0 = \inf_{v \in \mathcal{M}} J(v).$$

Thus, by Corollary 2.2, there exists a minimizing sequence (w_n) in \mathcal{M} , which is bounded, by Lemma 2.3. Hence, by Sobolev Imbedding Theorem, without loss of generality, we can assume up to a subsequence that there exist $w \in H_0^1(\Omega)$ such that

$$w_n \rightharpoonup w \text{ in } H_0^1(\Omega), \quad w_n \rightarrow w \text{ in } L^q(\Omega) \text{ with } q \in (1, 6) \text{ and } w_n(x) \rightarrow w(x) \text{ a.e in } \Omega.$$

Then, (f_2) combined with the *compactness lemma of Strauss* [7, Theorem A.I, p.338] gives

$$\begin{aligned} \lim_n \int_{\Omega_j} |w_{n,j}|^q dx &= \int_{\Omega_j} |w_j|^q dx, \\ \lim_n \int_{\Omega_j} w_{n,j} f(w_{n,j}) dx &= \int_{\Omega_j} w_j f(w_j) dx \end{aligned}$$

and

$$\lim_n \int_{\Omega_j} F(w_{n,j}) dx = \int_{\Omega_j} F(w_j) dx,$$

from where it follows together with Lemma 2.4 that $w_j \neq 0$ for $j = 1, 2$. Thereby, by Lemma 2.1, there are $t, s > 0$ verifying

$$J'(tw_1 + sw_2)w_1 = 0 \text{ and } J'(tw_1 + sw_2)w_2 = 0.$$

Now, let us prove that $t, s \leq 1$. First of all, we observe that subcritical growth of f leads to growth, we get

$$\int_{\Omega} f(w_{n,j})w_{n,j}dx \rightarrow \int_{\Omega} f(w_j)w_jdx$$

and

$$\int_{\Omega} F(w_{n,j})dx \rightarrow \int_{\Omega} F(w_j)dx.$$

Thus, as $J'(w_n)w_{n,j} = 0$,

$$M(\|w_n\|^2)\|w_{n,1}\|^2 = \int_{\Omega_1} f(w_{n,1})w_{n,1},$$

or equivalently,

$$\frac{M(\|w_n\|^2)}{\|w_n\|^2}\|w_{n,1}\|^2\|w_n\|^2 = \int_{\Omega_1} \frac{f(w_{n,1})}{w_{n,1}^3}w_{n,1}^4dx.$$

Taking the limit in the above equality, we find

$$\frac{M(\|w\|^2)}{\|w\|^2}\|w_1\|^2\|w\|^2 \leq \int_{\Omega_1} \frac{f(w_1)}{w_1^3}w_1^4dx. \quad (2.13)$$

On the other hand, as $J'(tw_1 + sw_2)tw_1 = 0$, we must have

$$M(\|tw_1 + sw_2\|^2)\|tw_1\|^2 = \int_{\Omega_1} f(tw_1)tw_1dx.$$

Without generality, we can suppose $s \leq t$. Hence,

$$\frac{M(t^2\|w\|^2)}{t^2\|w\|^2}\|w_1\|^2\|w\|^2 \geq \int_{\Omega_1} \frac{f(tw_1)}{(tw_1)^3}w_1^4dx. \quad (2.14)$$

Combining (2.13) with (2.14), we derive

$$\left[\frac{M(t^2\|w\|^2)}{t^2\|w\|^2} - \frac{M(\|w\|^2)}{\|w\|^2} \right] \|w_1\|^2\|w\|^2 \geq \int_{\Omega_1} \left[\frac{f(tw_1)}{(tw_1)^3} - \frac{f(w_1)}{(w_1)^3} \right] w_1^4dx.$$

Gathering (M_2) and (f_4) , we ensure that $0 < s \leq t \leq 1$.

In the next step, we will show that $J(tw_1 + sw_2) = c_0$. Since $tw_1 + sw_2 \in \mathcal{M}$ and $t, s \leq 1$, from (2.10) and (2.12),

$$\begin{aligned} c_0 \leq J(tw_1 + sw_2) &= J(tw_1 + sw_2) - \frac{1}{4}J'(tw_1 + sw_2)(tw_1 + sw_2) \\ &= \left[\frac{1}{2}\widehat{M}(\|tw_1 + sw_2\|^2) - \frac{1}{4}M(\|tw_1 + sw_2\|^2)\|tw_1 + sw_2\|^2 \right] \\ &+ \left[\int_{\Omega} \frac{1}{4}f(tw_1 + sw_2)(tw_1 + sw_2) - F(tw_1 + sw_2) \right] dx \\ &\leq \left[\frac{1}{2}\widehat{M}(\|w_1 + w_2\|^2) - \frac{1}{4}M(\|w_1 + w_2\|^2)\|w_1 + w_2\|^2 \right] \\ &+ \left[\int_{\Omega} \frac{1}{4}f(w_1 + w_2)(w_1 + w_2) - F(w_1 + w_2) \right] dx \leq \liminf_{n \rightarrow +\infty} J(w_n) = c_0. \end{aligned}$$

To complete the proof of Theorem 2.5, we claim that w is a critical point for functional J . To see why, for each $\varphi \in H_0^1(\Omega)$, we introduce the functions $Q^i : \mathbb{R}^3 \rightarrow \mathbb{R}$ given by

$$Q^1(r, z, l) = M(\|w + r\varphi + zw_1 + lw_2\|^2) \|w + r\varphi + zw_1\|_1^2 - \int_{\Omega_1} f(w_1 + r\varphi_1 + zw_1)(w_1 + r\varphi_1 + zw_1) dx.$$

and

$$Q^2(r, z, l) = M(\|w + r\varphi + zw_1 + lw_2\|^2) \|w + r\varphi + lw_2\|_2^2 - \int_{\Omega_2} f(w_2 + r\varphi_1 + lw_2)(w_2 + r\varphi_1 + lw_2) dx.$$

By a direct computation,

$$\frac{\partial Q^1}{\partial z}(0, 0, 0) = 2(M'(\|w\|^2) \|w_1\|^4 + M(\|w\|^2) \|w_1\|^2) - \int_{\Omega_1} (f'(w_1)w_1^2 + f(w_1)w_1) dx.$$

By inequality (2.9) and (2.11),

$$-\frac{\partial Q^1}{\partial z}(0, 0, 0) > 2M'(\|w\|^2) \|w_1\|^2 \|w_2\|^2.$$

Using a similar argument, it is possible to prove that

$$-\frac{\partial Q^2}{\partial l}(0, 0, 0) > 2M'(\|w\|^2) \|w_1\|^2 \|w_2\|^2 \quad \text{and} \quad \frac{\partial Q^1}{\partial l}(0, 0, 0) = \frac{\partial Q^2}{\partial z}(0, 0, 0) = M'(\|w\|^2) \|w_2\|^2 \|w_1\|^2.$$

From this,

$$\left| \begin{array}{cc} \frac{\partial Q^1}{\partial z}(0, 0, 0) & \frac{\partial Q^2}{\partial l}(0, 0, 0) \\ \frac{\partial Q^1}{\partial l}(0, 0, 0) & \frac{\partial Q^2}{\partial z}(0, 0, 0) \end{array} \right| = 3(M'(\|w\|^2))^2 \|w_2\|^4 \|w_1\|^4 > 0.$$

Therefore, applying the implicit function theorem, there are functions $z(r), l(r)$ of class C^1 defined on some interval $(-\delta, \delta)$, $\delta > 0$ such that $z(0) = l(0) = 0$ and

$$Q^i(r, z(r), l(r)) = 0, \quad r \in (-\delta, \delta), i = 1, 2.$$

This shows that for any $r \in (-\delta, \delta)$,

$$v(r) = w + r\varphi + z(r)w_1 + l(r)w_2 \in \mathcal{M}.$$

Then

$$J(v(r)) \geq J(w), \quad \forall r \in (-\delta, \delta),$$

that is,

$$J(w + r\varphi + z(r)w_1 + l(r)w_2) \geq J(w), \quad \forall r \in (-\delta, \delta).$$

From this,

$$\frac{J(w + r\varphi + z(r)w_1 + l(r)w_2) - J(w)}{r} \geq 0, \quad \forall r \in (0, \delta).$$

Taking the limit of $r \rightarrow 0$, we get

$$J'(w)(\varphi + z'(0)w_1 + l'(0)w_2) \geq 0.$$

Recalling that $J'(w)w_1 = J'(w)w_2 = 0$, the above inequality loads to

$$J'(w)\varphi \geq 0, \quad \forall \varphi \in H_0^1(\Omega)$$

and so,

$$J'(w)\varphi = 0, \quad \forall \varphi \in H_0^1(\Omega),$$

showing that w is a critical point for J . □

3 An auxiliary Kirchhoff problem

In this section, we work with an auxiliary problem adapting the ideas explored by del Pino & Felmer in [12] (see also [1] and [13]).

We start recalling that the energy functional $I_\lambda: E_\lambda \rightarrow \mathbb{R}$ associated with $(P)_\lambda$ is given by

$$I_\lambda(u) = \frac{1}{2} \widehat{M}(\|u\|_\lambda^2) - \int_{\mathbb{R}^3} F(u) dx,$$

where $E_\lambda = (E, \|\cdot\|_\lambda)$ with

$$E = \left\{ u \in H^1(\mathbb{R}^3); \int_{\mathbb{R}^3} a(x)|u|^2 dx < \infty \right\}$$

and

$$\|u\|_\lambda = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + (\lambda a(x) + 1)|u|^2) dx \right)^{\frac{1}{2}}.$$

By (a_0) , the embedding below

$$E_\lambda \hookrightarrow H^1(\mathbb{R}^3)$$

is continuous for all $\lambda \geq 0$. Consequently, E_λ is compactly embedded in $L_{loc}^s(\mathbb{R}^3)$, for all $1 \leq s < 6$. A direct computation gives that E_λ is a Hilbert space. Moreover, if $\mathcal{O} \subset \mathbb{R}^3$ is an open set, from the relation

$$\int_{\mathcal{O}} (|\nabla u|^2 + (\lambda a(x) + 1)|u|^2) dx \geq \int_{\mathcal{O}} |u|^2 dx, \quad \forall u \in E_\lambda \quad (3.1)$$

fixed $\delta \in (0, 1)$, there is $\nu > 0$, such that

$$\|u\|_{\lambda, \mathcal{O}}^2 - \nu \|u\|_{2, \mathcal{O}}^2 \geq \delta \|u\|_{\lambda, \mathcal{O}}^2, \quad \forall u \in E_\lambda, \lambda \geq 0, \quad (3.2)$$

where,

$$\|u\|_{\lambda, \mathcal{O}} = \left(\int_{\mathcal{O}} (|\nabla u|^2 + (\lambda a(x) + 1)|u|^2) dx \right)^{\frac{1}{2}}$$

and

$$\|u\|_{2, \mathcal{O}} = \left(\int_{\mathcal{O}} |u|^2 dx \right)^{\frac{1}{2}}.$$

We recall that given $\epsilon > 0$, the hypotheses (f_1) and (f_2) yield

$$|f(s)| \leq \epsilon |s| + C_\epsilon |s|^5, \quad \text{and } s \in \mathbb{R}. \quad (3.3)$$

Hence,

$$|F(s)| \leq \frac{\epsilon}{2} |s|^2 + \frac{C_\epsilon}{6} |s|^6, \quad \forall s \in \mathbb{R}, \quad (3.4)$$

where C_ϵ depends on ϵ . Moreover, for $\nu > 0$ fixed in (3.2), the assumptions (f_1) and (f_4) imply that there is a unique $\xi > 0$ verifying

$$\frac{f(\xi)}{\xi} = \nu \quad (3.5)$$

Using the numbers ξ and ν , we set the function $\tilde{f}: \mathbb{R} \rightarrow \mathbb{R}$ given by

$$\tilde{f}(s) = \begin{cases} f(s), & s \leq \xi \\ \nu s, & s \geq \xi \end{cases},$$

which fulfills the inequality

$$\tilde{f}(s) \leq \nu|s|, \quad \forall s \in \mathbb{R}. \quad (3.6)$$

Thus

$$\tilde{f}(s)s \leq \nu|s|^2, \quad \forall s \in \mathbb{R} \quad (3.7)$$

and

$$\tilde{F}(s) \leq \frac{\nu}{2}|s|^2, \quad \forall s \in \mathbb{R}, \quad (3.8)$$

where $\tilde{F}(s) = \int_0^s \tilde{f}(t) dt$.

Now, since $\Omega = \text{int}(a^{-1}(\{0\}))$ is formed by k connected components $\Omega_1, \dots, \Omega_k$ with $\text{dist}(\Omega_i, \Omega_j) > 0$, $i \neq j$, then for each $j \in \{1, \dots, k\}$, we are able to fix a smooth bounded domain Ω'_j such that

$$\overline{\Omega_j} \subset \Omega'_j \text{ and } \overline{\Omega'_i} \cap \overline{\Omega'_j} = \emptyset, \text{ for } i \neq j. \quad (3.9)$$

From now on, we fix a non-empty subset $\Upsilon \subset \{1, \dots, k\}$,

$$\Omega_\Upsilon = \bigcup_{j \in \Upsilon} \Omega_j, \quad \Omega'_\Upsilon = \bigcup_{j \in \Upsilon} \Omega'_j \text{ and } \chi_\Upsilon = \begin{cases} 1, & \text{if } x \in \Omega'_\Upsilon \\ 0, & \text{if } x \notin \Omega'_\Upsilon. \end{cases}$$

Using the above notations, we set the functions

$$g(x, s) = \chi_\Upsilon(x)f(s) + (1 - \chi_\Upsilon(x))\tilde{f}(s), \quad (x, s) \in \mathbb{R}^3 \times \mathbb{R}$$

and

$$G(x, s) = \int_0^s g(x, t) dt, \quad (x, s) \in \mathbb{R}^3 \times \mathbb{R},$$

and the auxiliary Kirchhoff problem

$$(A_\lambda) \quad \begin{cases} M(\|u\|_\lambda^2) \left(-\Delta u + (\lambda a(x) + 1)u \right) = g(x, u), & \text{in } \mathbb{R}^3, \\ u \in E_\lambda. \end{cases}$$

The problem (A_λ) is strongly related to (P_λ) , in the sense that, if u_λ is a solution for (A_λ) verifying

$$u_\lambda(x) \leq \xi, \quad \forall x \in \mathbb{R}^N \setminus \Omega'_\Upsilon,$$

then it is a solution for (P_λ) .

In comparison to (P_λ) , problem (A_λ) has the advantage that the energy functional associated with (A_λ) , namely, $\phi_\lambda: E_\lambda \rightarrow \mathbb{R}$ given by

$$\phi_\lambda(u) = \frac{1}{2} \widehat{M}(\|u\|_\lambda^2) - \int_{\mathbb{R}^3} G(x, u) dx,$$

satisfies the (PS) condition, whereas I_λ does not necessarily satisfy this condition.

Proposition 3.1. All $(PS)_d$ sequences for ϕ_λ are bounded in E_λ .

Proof. Let (u_n) be a $(PS)_d$ sequence for ϕ_λ . So, there is $n_0 \in \mathbb{N}$ such that

$$\phi_\lambda(u_n) - \frac{1}{\theta} \phi'_\lambda(u_n) u_n \leq d + 1 + \|u_n\|_\lambda, \text{ for } n \geq n_0.$$

On the other hand, by (3.7) and (3.8),

$$\tilde{F}(s) - \frac{1}{\theta} \tilde{f}(s) s \leq \left(\frac{1}{2} - \frac{1}{\theta} \right) \nu |s|^2, \forall s \in \mathbb{R}^3, s \in \mathbb{R},$$

which together with (3.2) gives

$$\phi_\lambda(u_n) - \frac{1}{\theta} \phi'_\lambda(u_n) u_n \geq \left(\frac{1}{2} - \frac{1}{\theta} \right) \delta \|u_n\|_\lambda^2, \forall n \in \mathbb{N},$$

from where it follows that (u_n) is bounded in E_λ . \square

Proposition 3.2. If (u_n) is a $(PS)_d$ sequence for ϕ_λ , then given $\epsilon > 0$, there is $R > 0$ such that

$$\limsup_n \int_{\mathbb{R}^3 \setminus B_R(0)} (|\nabla u_n|^2 + (\lambda a(x) + 1) |u_n|^2) dx < \epsilon. \quad (3.10)$$

Hence, once that g has a subcritical growth, if $u \in E_\lambda$ is the weak limit of (u_n) , then

$$\int_{\mathbb{R}^3} g(x, u_n) u_n dx \rightarrow \int_{\mathbb{R}^3} g(x, u) u dx \quad \text{and} \quad \int_{\mathbb{R}^3} g(x, u_n) v dx \rightarrow \int_{\mathbb{R}^3} g(x, u) v dx, \forall v \in E_\lambda.$$

Proof. Let (u_n) be a $(PS)_d$ sequence for ϕ_λ , $R > 0$ large such that $\Omega'_\Gamma \subset B_{\frac{R}{2}}(0)$ and $\eta_R \in C^\infty(\mathbb{R}^3)$ satisfying

$$\eta_R(x) = \begin{cases} 0, & x \in B_{\frac{R}{2}}(0) \\ 1, & x \in \mathbb{R}^3 \setminus B_R(0) \end{cases},$$

$0 \leq \eta_R \leq 1$ and $|\nabla \eta_R| \leq \frac{C}{R}$, where $C > 0$ does not depend on R . This way,

$$\begin{aligned} m_0 \|u_n \eta_R\|_\lambda^2 &\leq \int_{\mathbb{R}^3} M(\|u_n\|_\lambda^2) (|\nabla u_n|^2 + (\lambda a(x) + 1) |u_n|^2) \eta_R dx \\ &= \phi'_\lambda(u_n) (u_n \eta_R) - \int_{\mathbb{R}^3} M(\|u_n\|_\lambda^2) u_n \nabla u_n \nabla \eta_R dx + \int_{\mathbb{R}^3 \setminus \Omega'_\Gamma} \tilde{f}(u_n) u_n \eta_R dx. \end{aligned}$$

Denoting

$$L = m_0 \int_{\mathbb{R}^3} (|\nabla u_n|^2 + (\lambda a(x) + 1) |u_n|^2) \eta_R dx,$$

it follows from (3.7),

$$L \leq \phi'_\lambda(u_n) (u_n \eta_R) + \frac{C}{R} \int_{\mathbb{R}^3} M(\|u_n\|_\lambda^2) |u_n| |\nabla u_n| dx + \nu \int_{\mathbb{R}^3} |u_n|^2 \eta_R dx.$$

The Hölder's inequality in conjunction with the boundedness of (u_n) and $(|\nabla u_n|)$ in $L^2(\mathbb{R}^3)$, ensures that

$$L \leq o_n(1) + \frac{C}{(1 - \nu)R}.$$

Therefore

$$\limsup_n \int_{\mathbb{R}^3 \setminus B_R(0)} m_0(|\nabla u_n|^2 + (\lambda a(x) + 1)|u_n|^2) dx \leq \frac{C}{(1-\nu)R}.$$

So, given $\epsilon > 0$, choosing a $R > 0$ possibly still bigger, we have that $\frac{C}{(1-\nu)R} < \epsilon$, which proves (3.10). Now, we will show that

$$\int_{\mathbb{R}^3} g(x, u_n) u_n dx \rightarrow \int_{\mathbb{R}^3} g(x, u) u dx.$$

Using the fact that $g(x, u)u \in L^1(\mathbb{R}^3)$ together with (3.10) and Sobolev embeddings, given $\epsilon > 0$, we can choose $R > 0$ such that

$$\limsup_{n \rightarrow +\infty} \int_{\mathbb{R}^3 \setminus B_R(0)} |g(x, u_n) u_n| dx \leq \frac{\epsilon}{4} \quad \text{and} \quad \int_{\mathbb{R}^3 \setminus B_R(0)} |g(x, u) u| dx \leq \frac{\epsilon}{4}.$$

On the other hand, since g has a subcritical growth, the compact Sobolev embeddings lead to

$$\int_{B_R(0)} g(x, u_n) u_n dx \rightarrow \int_{B_R(0)} g(x, u) u dx.$$

Combining the above information, we conclude that

$$\int_{\mathbb{R}^3} g(x, u_n) u_n dx \rightarrow \int_{\mathbb{R}^3} g(x, u) u dx.$$

The same type of argument works to prove that

$$\int_{\mathbb{R}^3} g(x, u_n) v dx \rightarrow \int_{\mathbb{R}^3} g(x, u) v dx, \quad \forall v \in E_\lambda.$$

□

The next result does not appear in [13], however since we are working with the Kirchhoff problem type, it is required here.

Proposition 3.3. *Let (u_n) be a $(PS)_d$ sequence for ϕ_λ such that $u_n \rightharpoonup u$, then*

$$\lim_{n \rightarrow \infty} \int_{B_R} [|\nabla u_n|^2 + (\lambda a(x) + 1)u_n^2] dx = \int_{B_R} [|\nabla u|^2 + (\lambda a(x) + 1)u^2] dx,$$

for all $R > 0$.

Proof. We can assume that $\|u_n\|_\lambda \rightarrow t_0$, thus, we have $\|u\|_\lambda \leq t_0$. Let $\eta_\rho \in C^\infty(\mathbb{R}^3)$ such that

$$\eta_\rho(x) = \begin{cases} 1 & \text{se } x \in B_\rho(0) \\ 0 & \text{se } x \notin B_{2\rho}(0). \end{cases}$$

with $0 \leq \eta_\rho(x) \leq 1$. Let,

$$P_n(x) = M(\|u_n\|_\lambda^2) [|\nabla u_n - \nabla u|^2 + (\lambda a(x) + 1)(u_n - u)^2].$$

For each $R > 0$ fixed, choosing $\rho > R$ we obtain

$$\int_{B_R} P_n = \int_{B_R} P_n \eta_\rho \leq M(\|u_n\|_\lambda^2) \int_{\mathbb{R}^3} [|\nabla u_n - \nabla u|^2 + (\lambda a(x) + 1)(u_n - u)^2] \eta_\rho.$$

By expanding the inner product in \mathbb{R}^3 ,

$$\begin{aligned} \int_{B_R} P_n &\leq M(\|u_n\|_\lambda^2) \int_{\mathbb{R}^3} [|\nabla u_n|^2 + (\lambda a(x) + 1)(u_n)^2] \eta_\rho \\ &\quad - 2M(\|u_n\|_\lambda^2) \int_{\mathbb{R}^3} [\nabla u_n \nabla u + (\lambda a(x) + 1)u_n u] \eta_\rho \\ &\quad + M(\|u_n\|_\lambda^2) \int_{\mathbb{R}^3} [|\nabla u|^2 + (\lambda a(x) + 1)u^2] \eta_\rho. \end{aligned}$$

Setting

$$I_{n,\rho}^1 = M(\|u_n\|_\lambda^2) \int_{\mathbb{R}^3} [|\nabla u_n|^2 + (\lambda a(x) + 1)(u_n)^2] \eta_\rho - \int_{\mathbb{R}^3} g(x, u_n) u_n \eta_\rho,$$

$$I_{n,\rho}^2 = M(\|u_n\|_\lambda^2) \int_{\mathbb{R}^3} [\nabla u_n \nabla u + (\lambda a(x) + 1)u_n u] \eta_\rho - \int_{\mathbb{R}^3} g(x, u_n) u \eta_\rho,$$

$$I_{n,\rho}^3 = -M(\|u_n\|_\lambda^2) \int_{\mathbb{R}^3} [\nabla u_n \nabla u + (\lambda a(x) + 1)u_n u] \eta_\rho + M(\|u_n\|_\lambda^2) \int_{\mathbb{R}^3} [|\nabla u|^2 + (\lambda a(x) + 1)u^2] \eta_\rho$$

and

$$I_{n,\rho}^4 = \int_{\mathbb{R}^3} g(x, u_n) u_n \eta_\rho - \int_{\mathbb{R}^3} g(x, u_n) u \eta_\rho,$$

we find the estimate below

$$0 \leq \int_{B_R} P_n \leq |I_{n,\rho}^1| + |I_{n,\rho}^2| + |I_{n,\rho}^3| + |I_{n,\rho}^4|. \quad (3.11)$$

Observe that

$$I_{n,\rho}^1 = \phi'_\lambda(u_n)(u_n \eta_\rho) - M(\|u_n\|_\lambda^2) \int_{\mathbb{R}^3} u_n \nabla u_n \nabla \eta_\rho.$$

Recalling that $(u_n \eta_\rho)$ is bounded in H_ε , we have $\phi'_\lambda(u_n)(u_n \eta_\rho) = o_n(1)$. Moreover, from a straightforward computation

$$\lim_{\rho \rightarrow \infty} \left[\limsup_{n \rightarrow \infty} \left| M(\|u_n\|_\lambda^2) \int_{\mathbb{R}^3} u_n \nabla u_n \nabla \eta_\rho \right| \right] = 0.$$

Then,

$$\lim_{\rho \rightarrow \infty} \left[\limsup_{n \rightarrow \infty} |I_{n,\rho}^1| \right] = 0. \quad (3.12)$$

We also see that

$$I_{n,\rho}^2 = \phi'_\lambda(u_n)(u \eta_\rho) - M(\|u_n\|_\lambda^2) \int_{\mathbb{R}^3} u \nabla u_n \nabla \eta_\rho.$$

By arguing in the same way as in the previous case,

$$\phi'_\lambda(u_n)(u \eta_\rho) = o_n(1)$$

and

$$\lim_{\rho \rightarrow \infty} \left[\limsup_{n \rightarrow \infty} \left| M(\|u_n\|_\lambda^2) \int_{\mathbb{R}^3} u \nabla u_n \nabla \eta_\rho \right| \right] = 0.$$

Therefore,

$$\lim_{\rho \rightarrow \infty} \left[\limsup_{n \rightarrow \infty} |I_{n,\rho}^2| \right] = 0. \quad (3.13)$$

On the other hand, from the weak convergence

$$\lim_{n \rightarrow \infty} |I_{n,\rho}^3| = 0, \quad \forall \rho > R. \quad (3.14)$$

Finally, from

$$u_n \rightharpoonup u, \text{ in } L_{loc}^s(\mathbb{R}^3), 1 \leq s < 6,$$

we conclude that

$$\lim_{n \rightarrow \infty} |I_{n,\rho}^4| = 0, \quad \forall \rho > R. \quad (3.15)$$

From (3.11), (3.12), (3.13), (3.14) and (3.15), we obtain

$$0 \leq \limsup_{n \rightarrow \infty} \int_{B_R} P_n \leq 0,$$

or equivalently

$$\lim_{n \rightarrow \infty} \int_{B_R} P_n = 0.$$

Therefore,

$$\lim_{n \rightarrow \infty} \int_{B_R} [|\nabla u_n|^2 + (\lambda a(x) + 1)u_n^2] = \int_{B_R} [|\nabla u|^2 + (\lambda a(x) + 1)u^2].$$

□

Proposition 3.4. ϕ_λ verifies the (PS) condition.

Proof Let (u_n) be a $(PS)_d$ sequence for ϕ_λ and $u \in E_\lambda$ such that $u_n \rightharpoonup u$ in E_λ . Thereby, by Proposition 3.2,

$$\int_{\mathbb{R}^3} g(x, u_n) u_n dx \rightarrow \int_{\mathbb{R}^3} g(x, u) u dx \quad \text{and} \quad \int_{\mathbb{R}^3} g(x, u_n) v dx \rightarrow \int_{\mathbb{R}^3} g(x, u) v dx, \quad \forall v \in E_\lambda.$$

Moreover, the weak limit also gives

$$\int_{\mathbb{R}^3} \nabla u \nabla (u_n - u) dx \rightarrow 0$$

and

$$\int_{\mathbb{R}^3} (\lambda a(x) + 1) u (u_n - u) dx \rightarrow 0.$$

Gathering $\phi'_\lambda(u_n)u_n = o_n(1)$, $\phi'_\lambda(u_n)u = o_n(1)$, (M_1) and the above limits, we derive that

$$\|u_n - u\|_\lambda^2 \rightarrow 0,$$

finishing the proof. □

4 The $(PS)_\infty$ condition

A sequence $(u_n) \subset H^1(\mathbb{R}^3)$ is called a $(PS)_\infty$ sequence for the family $(\phi_\lambda)_{\lambda \geq 1}$, if there is a sequence $(\lambda_n) \subset [1, \infty)$ with $\lambda_n \rightarrow \infty$, as $n \rightarrow \infty$, verifying

$$\phi_{\lambda_n}(u_n) \rightarrow c \text{ and } \|\phi'_{\lambda_n}(u_n)\|_{E_{\lambda_n}^*} \rightarrow 0, \text{ as } n \rightarrow \infty,$$

for some $c \in \mathbb{R}$.

Proposition 4.1. *Let $(u_n) \subset H^1(\mathbb{R}^3)$ be a $(PS)_\infty$ sequence for $(\phi_\lambda)_{\lambda \geq 1}$. Then, up to a subsequence, there exists $u \in H^1(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$. Furthermore,*

$$(i) \quad u_n \rightarrow u \text{ in } H^1(\mathbb{R}^3);$$

$$(ii) \quad u = 0 \text{ in } \mathbb{R}^3 \setminus \Omega_\Upsilon, \quad u|_{\Omega_j} \geq 0 \text{ for all } j \in \Upsilon, \text{ and } u \text{ is a solution for}$$

$$\begin{cases} M \left(\int_{\Omega_\Upsilon} (|\nabla u|^2 + u^2) dx \right) (-\Delta u + u) = f(u), & \text{in } \Omega_\Upsilon, \\ u \in H_0^1(\Omega_\Upsilon); \end{cases} \quad (P)_{\infty, \Upsilon}$$

$$(iii) \quad \lambda_n \int_{\mathbb{R}^3} a(x) |u_n|^2 dx \rightarrow 0;$$

$$(iv) \quad \|u_n - u\|_{\lambda, \Omega'_\Upsilon}^2 \rightarrow 0, \text{ for } j \in \Upsilon;$$

$$(v) \quad \|u_n\|_{\lambda, \mathbb{R}^3 \setminus \Omega'_\Upsilon}^2 \rightarrow 0;$$

$$(vi) \quad \phi_{\lambda_n}(u_n) \rightarrow \frac{1}{2} \widehat{M} \left(\int_{\Omega_\Upsilon} (|\nabla u|^2 + |u|^2) dx \right) - \int_{\Omega_\Upsilon} F(u) dx.$$

Proof. Using the Proposition 3.1, we know that $(\|u_n\|_{\lambda_n})$ is bounded in \mathbb{R} and (u_n) is bounded in $H^1(\mathbb{R}^3)$. So, up to a subsequence, there exists $u \in H^1(\mathbb{R}^3)$ such that

$$u_n \rightharpoonup u \text{ in } H^1(\mathbb{R}^3) \text{ and } u_n(x) \rightarrow u(x) \text{ for a.e. } x \in \mathbb{R}^3.$$

Now, for each $m \in \mathbb{N}$, we define $C_m = \left\{ x \in \mathbb{R}^3; a(x) \geq \frac{1}{m} \right\}$. Without loss of generality, we can assume $\lambda_n < 2(\lambda_n - 1)$, $\forall n \in \mathbb{N}$. Thus

$$\int_{C_m} |u_n|^2 dx \leq \frac{2m}{\lambda_n} \int_{C_m} (\lambda_n a(x) + 1) |u_n|^2 dx \leq \frac{C}{\lambda_n}.$$

By Fatou's lemma,

$$\int_{C_m} |u|^2 dx = 0,$$

implying that $u = 0$ in C_m , and so, $u = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$. From this, we are able to prove (i) – (vi).

(i) Since $u = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$, repeating the argument explored in Proposition 3.4, we get

$$\int_{\mathbb{R}^3} (|\nabla u_n - \nabla u|^2 + (\lambda_n a(x) + 1)|u_n - u|^2) dx \rightarrow 0,$$

which implies $u_n \rightarrow u$ in $H^1(\mathbb{R}^3)$.

(ii) Since $u \in H^1(\mathbb{R}^3)$ and $u = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$, we have $u \in H_0^1(\Omega)$ or, equivalently, $u|_{\Omega_j} \in H_0^1(\Omega_j)$, for $j = 1, \dots, k$. Moreover, the limit $u_n \rightarrow u$ in $H^1(\mathbb{R}^3)$ combined with $\phi'_{\lambda_n}(u_n)\varphi \rightarrow 0$ for $\varphi \in C_0^\infty(\Omega_\Upsilon)$ implies that

$$M\left(\int_{\Omega_\Upsilon} (|\nabla u|^2 + u^2) dx\right) \int_{\Omega_\Upsilon} (\nabla u \nabla \varphi + u \varphi) dx - \int_{\Omega_\Upsilon} f(u) \varphi dx = 0, \quad (4.1)$$

showing that $u|_{\Omega_\Upsilon}$ is a solution for the nonlocal problem

$$\begin{cases} M\left(\int_{\Omega_\Upsilon} (|\nabla u|^2 + u^2) dx\right) (-\Delta u + u) = f(u), & \text{in } \Omega_\Upsilon, \\ u \in H_0^1(\Omega_\Upsilon). \end{cases} \quad (P)_{\infty, \Upsilon}$$

On the other hand, if $j \notin \Upsilon$, we must have

$$M\left(\int_{\Omega_\Upsilon} (|\nabla u|^2 + u^2) dx\right) \int_{\Omega_j} (|\nabla u|^2 + |u|^2) dx - \int_{\Omega_j} \tilde{f}(u) u dx = 0.$$

The above equality combined with (3.7) and (3.2) gives

$$0 \geq \|u\|_{\lambda, \Omega_j}^2 - \nu \|u\|_{2, \Omega_j}^2 \geq \delta \|u\|_{\lambda, \Omega_j}^2(u) \geq 0,$$

from where it follows $u|_{\Omega_j} = 0$ for $j \notin \Upsilon$. This proves $u = 0$ outside Ω_Υ and $u \geq 0$ in \mathbb{R}^3 .

(iii) From (i),

$$\lambda_n \int_{\mathbb{R}^3} a(x) |u_n|^2 dx = \int_{\mathbb{R}^2} \lambda_n a(x) |u_n - u|^2 dx \leq \|u_n - u\|_{\lambda_n}^2,$$

loading to

$$\lambda_n \int_{\mathbb{R}^3} a(x) |u_n|^2 dx \rightarrow 0.$$

(iv) Let $j \in \Upsilon$. From (i),

$$|u_n - u|_{2, \Omega'_j}^2, |\nabla u_n - \nabla u|_{2, \Omega'_j}^2 \rightarrow 0.$$

Then,

$$\int_{\Omega'_\Upsilon} (|\nabla u_n|^2 - |\nabla u|^2) dx \rightarrow 0 \quad \text{and} \quad \int_{\Omega'_\Upsilon} (|u_n|^2 - |u|^2) dx \rightarrow 0.$$

From (iii),

$$\int_{\Omega'_\Upsilon} \lambda_n a(x) |u_n|^2 dx \rightarrow 0.$$

This way

$$\|u_n\|_{\lambda_n, \Omega'_\Upsilon}^2 \rightarrow \int_{\Omega_\Upsilon} (|\nabla u|^2 + |u|^2) dx.$$

(v) By (i), $\|u_n - u\|_{\lambda_n}^2 \rightarrow 0$, and so,

$$\|u_n\|_{\lambda_n, \mathbb{R}^3 \setminus \Omega_\Upsilon}^2 \rightarrow 0.$$

(vi) From (i) – (v),

$$\widehat{M}\left(\|u_n\|_{\lambda_n}^2\right) \rightarrow \widehat{M}\left(\int_{\Omega_\Upsilon} (|\nabla u|^2 + |u|^2) dx\right)$$

and

$$\int_{\mathbb{R}^3} G(x, u_n) dx \rightarrow \int_{\Omega_\Upsilon} F(u) dx.$$

Therefore

$$\phi_{\lambda_n}(u_n) \rightarrow \frac{1}{2} \widehat{M}\left(\int_{\Omega_\Upsilon} (|\nabla u|^2 + |u|^2) dx\right) - \int_{\Omega_\Upsilon} F(u) dx.$$

□

5 The boundedness of the (A_λ) solutions

In this section, we study the boundedness outside Ω'_Υ for some solutions of (A_λ) . To this end, we adapt the arguments found in [1] and [19] for our new setting.

Proposition 5.1. *Let (u_λ) be a family of solutions for (A_λ) such that $u_\lambda \rightarrow 0$ in $H^1(\mathbb{R}^3 \setminus \Omega_\Upsilon)$, as $\lambda \rightarrow \infty$. Then, there exists $\lambda^* > 0$ with the following property:*

$$|u_\lambda|_{\infty, \mathbb{R}^3 \setminus \Omega'_\Upsilon} \leq \xi, \quad \forall \lambda \geq \lambda^*.$$

Hence, u_λ is a solution for (P_λ) for $\lambda \geq \lambda^*$.

Proof. Since $\partial\Omega'_\Upsilon$ is a compact set, fixed a neighborhood \mathcal{B} of $\partial\Omega'_\Upsilon$ such that

$$\mathcal{B} \subset \mathbb{R}^3 \setminus \Omega_\Upsilon,$$

the iteration Moser technique implies that there is $C > 0$, which is independent of λ , such that

$$|u_\lambda|_{L^\infty(\partial\Omega'_\Upsilon)} \leq C|u_\lambda|_{L^{2^*}(\mathcal{B})}$$

Since $u_\lambda \rightarrow 0$ in $H^1(\mathbb{R}^3 \setminus \Omega_\Upsilon)$, we have that

$$|u_\lambda|_{L^{2^*}(\mathcal{B})} \rightarrow 0 \text{ as } \lambda \rightarrow \infty.$$

Hence, there is $\lambda^* > 0$ such that

$$|u_\lambda|_{L^{2^*}(\mathcal{B})} < \frac{\xi}{2C} \quad \forall \lambda \geq \lambda^*,$$

and so,

$$|u_\lambda|_{L^\infty(\partial\Omega'_\Upsilon)} < \xi \quad \forall \lambda \geq \lambda^*.$$

Next, for $\lambda \geq \lambda^*$, we set $\tilde{u}_\lambda : \mathbb{R}^3 \setminus \Omega'_\Upsilon \rightarrow \mathbb{R}$ given by

$$\tilde{u}_\lambda(x) = (u_\lambda - \xi)^+(x).$$

Thereby, $\tilde{u}_\lambda \in H_0^1(\mathbb{R}^3 \setminus \Omega'_\Upsilon)$. Our goal is showing that $\tilde{u}_\lambda = 0$ in $\mathbb{R}^3 \setminus \Omega'_\Upsilon$, because this will imply that

$$|u_\lambda|_{\infty, \mathbb{R}^3 \setminus \Omega'_\Upsilon} \leq \xi.$$

In fact, extending $\tilde{u}_\lambda = 0$ in Ω'_Υ and taking \tilde{u}_λ as a test function and using (M_1) , we obtain

$$m_0 \left(\int_{\mathbb{R}^3 \setminus \Omega'_\Upsilon} \nabla u_\lambda \nabla \tilde{u}_\lambda dx + \int_{\mathbb{R}^3 \setminus \Omega'_\Upsilon} (\lambda a(x) + 1) u_\lambda \tilde{u}_\lambda dx \right) \leq \int_{\mathbb{R}^N \setminus \Omega'_\Upsilon} g(x, u_\lambda) \tilde{u}_\lambda dx.$$

Since

$$\begin{aligned} \int_{\mathbb{R}^3 \setminus \Omega'_\Upsilon} \nabla u_\lambda \nabla \tilde{u}_\lambda dx &= \int_{\mathbb{R}^3 \setminus \Omega'_\Upsilon} |\nabla \tilde{u}_\lambda|^2 dx, \\ \int_{\mathbb{R}^3 \setminus \Omega'_\Upsilon} (\lambda a(x) + 1) u_\lambda \tilde{u}_\lambda dx &= \int_{(\mathbb{R}^3 \setminus \Omega'_\Upsilon)_+} (\lambda a(x) + 1) (\tilde{u}_\lambda + \xi) \tilde{u}_\lambda dx \end{aligned}$$

and

$$\int_{\mathbb{R}^3 \setminus \Omega'_\Upsilon} g(x, u_\lambda) \tilde{u}_\lambda dx = \int_{(\mathbb{R}^3 \setminus \Omega'_\Upsilon)_+} \frac{g(x, u_\lambda)}{u_\lambda} (\tilde{u}_\lambda + \xi) \tilde{u}_\lambda dx,$$

where

$$(\mathbb{R}^3 \setminus \Omega'_\Upsilon)_+ = \{x \in \mathbb{R}^3 \setminus \Omega'_\Upsilon; u_\lambda(x) > \xi\},$$

we derive

$$m_0 \left(\int_{\mathbb{R}^3 \setminus \Omega'_\Upsilon} |\nabla \tilde{u}_\lambda|^2 dx + \int_{(\mathbb{R}^3 \setminus \Omega'_\Upsilon)_+} ((\lambda a(x) + 1) - \frac{g(x, u_\lambda)}{u_\lambda}) (\tilde{u}_\lambda + \xi) \tilde{u}_\lambda dx \right) \leq 0,$$

Now, by (3.6),

$$(\lambda a(x) + 1) - \frac{g(x, u_\lambda)}{u_\lambda} > \nu - \frac{\tilde{f}(x, u_\lambda)}{u_\lambda} \geq 0 \quad \text{in } (\mathbb{R}^3 \setminus \Omega'_\Upsilon)_+.$$

This form, $\tilde{u}_\lambda = 0$ in $(\mathbb{R}^3 \setminus \Omega'_\Upsilon)_+$. Obviously, $\tilde{u}_\lambda = 0$ at the points where $u_\lambda \leq \xi$, consequently, $\tilde{u}_\lambda = 0$ in $\mathbb{R}^N \setminus \Omega'_\Upsilon$. \square

6 A special minimax value for ϕ_λ

For fixed non-empty subset $\Upsilon \subset \{1, \dots, k\}$, consider

$$I_\Upsilon(u) = \frac{1}{2} \widehat{M} \left(\int_{\Omega_\Upsilon} (|\nabla u|^2 + |u|^2) dx \right) - \int_{\Omega_\Upsilon} F(u) dx, \quad u \in H_0^1(\Omega_\Upsilon),$$

the energy functional associated to $(P)_{\infty, \Upsilon}$, and $\phi_{\lambda, \Upsilon} : H^1(\Omega'_\Upsilon) \rightarrow \mathbb{R}$ given by

$$\phi_{\lambda, \Upsilon}(u) = \frac{1}{2} \widehat{M} \left(\int_{\Omega'_\Upsilon} (|\nabla u|^2 + (\lambda a(x) + 1)|u|^2) dx \right) - \int_{\Omega'_\Upsilon} F(u) dx,$$

the energy functional associated to the Kirchhoff problem

$$\begin{cases} M \left(\int_{\Omega'_\Upsilon} (|\nabla u|^2 + (\lambda a(x) + 1)|u|^2) dx \right) (-\Delta u + (\lambda a(x) + 1)u) = f(u), & \text{in } \Omega'_\Upsilon, \\ \frac{\partial u}{\partial \eta} = 0, & \text{on } \partial\Omega'_\Upsilon. \end{cases}$$

In the following, we denote by c_Υ the number given by

$$c_\Upsilon = \inf_{u \in \mathcal{M}_\Upsilon} I_\Upsilon(u)$$

where

$$\mathcal{M}_\Upsilon = \{u \in \mathcal{N}_\Upsilon : I'_\Upsilon(u)u_j = 0 \text{ and } u_j \neq 0 \ \forall j \in \Upsilon\}$$

with $u_j = u|_{\Omega_j}$ and

$$\mathcal{N}_\Upsilon = \{u \in H_0^1(\Omega_\Upsilon) \setminus \{0\} : I'_\Upsilon(u)u = 0\}.$$

Of a similar way, we denote by $c_{\lambda,\Upsilon}$ the number given by

$$c_{\lambda,\Upsilon} = \inf_{u \in \mathcal{M}'_\Upsilon} \phi_{\lambda,\Upsilon}(u)$$

where

$$\mathcal{M}'_\Upsilon = \{u \in \mathcal{N}'_\Upsilon : \phi'_{\lambda,\Upsilon}(u)\tilde{u}_j = 0 \text{ and } \tilde{u}_j \neq 0 \ \forall j \in \Upsilon\}$$

with

$$\tilde{u}_j(x) = \begin{cases} u(x), & x \in \Omega'_j \\ 0, & x \in \Omega'_\Upsilon \setminus \Omega'_j \end{cases}$$

and

$$\mathcal{N}'_\Upsilon = \{u \in H^1(\Omega'_\Upsilon) \setminus \{0\} : \phi'_{\lambda,\Upsilon}(u)u = 0\}.$$

Repeating the same approach used in Section 2, we ensure that there exist $w_\Upsilon \in H_0^1(\Omega_\Upsilon)$ and $w_{\lambda,\Upsilon} \in H^1(\Omega'_\Upsilon)$ such that

$$I_\Upsilon(w_\Upsilon) = c_\Upsilon \text{ and } I'_\Upsilon(w_\Upsilon) = 0$$

and

$$\phi_{\lambda,\Upsilon}(w_{\lambda,\Upsilon}) = c_{\lambda,\Upsilon} \text{ and } \phi'_{\lambda,\Upsilon}(w_{\lambda,\Upsilon}) = 0.$$

By a direct computation, it is possible to show that there is $\tau > 0$ such that if $u \in \mathcal{M}_\Upsilon$, then

$$\|u_j\|_j > \tau, \ \forall j \in \Upsilon, \quad (6.1)$$

where, $\|\cdot\|_j$ denotes the norm on $H_0^1(\Omega_j)$ given by

$$\|u\|_j = \left(\int_{\Omega_j} (|\nabla u|^2 + |u|^2) dx \right)^{\frac{1}{2}}.$$

In particular, since $w_\Upsilon \in \mathcal{M}_\Upsilon$, we also have

$$\|w_{\Upsilon,j}\|_j > \tau \ \forall j \in \Upsilon, \quad (6.2)$$

where $w_{\Upsilon,j} = w_{\Upsilon}|_{\Omega_j}$ for all $j \in \Upsilon$. Moreover, reviewing the proof of Theorem 2.5, it is possible to see that

$$I_{\Upsilon}(w_{\Upsilon}) = c_{\Upsilon} = \max\{I_{\Upsilon}(t_1 w_1 + \dots + t_l w_l) : (t_1, \dots, t_l) \in [0, +\infty)^l\} \quad (6.3)$$

and

$$I_{\Upsilon}(w_{\Upsilon}) = I_{\Upsilon}(w_1 + \dots + w_l) > I_{\Upsilon}(t_1 w_1 + \dots + t_l w_l), \quad \forall (t_1, \dots, t_l) \in [0, +\infty)^l \setminus \{(1, \dots, 1)\}. \quad (6.4)$$

Lemma 6.1. *There holds that*

$$(i) \quad 0 < c_{\lambda, \Upsilon} \leq c_{\Upsilon}, \quad \forall \lambda \geq 0;$$

$$(ii) \quad c_{\lambda, \Upsilon} \rightarrow c_{\Upsilon}, \quad \text{as } \lambda \rightarrow \infty.$$

Proof

(i) Using the inclusion $H_0^1(\Omega_{\Upsilon}) \subset H^1(\Omega'_{\Upsilon})$, it is easy to observe that

$$c_{\lambda, \Upsilon} \leq c_{\Upsilon}.$$

(ii) Let (λ_n) be such a sequence with $\lambda_n \rightarrow +\infty$ and consider an arbitrary subsequence of $(c_{\lambda_n, \Upsilon})$ (not relabelled). Let $w_n \in H^1(\Omega'_j)$ with

$$\phi_{\lambda_n, \Upsilon}(w_n) = c_{\lambda_n, \Upsilon} \quad \text{and} \quad \phi'_{\lambda_n, \Upsilon}(w_n) = 0.$$

By the previous item, $(c_{\lambda_n, \Upsilon})$ is bounded. Then, there exists (w_{n_k}) subsequence of (w_n) such that $(\phi_{\lambda_{n_k}, \Upsilon}(w_{n_k}))$ converges and $\phi'_{\lambda_{n_k}, \Upsilon}(w_{n_k}) = 0$. Now, repeating the arguments explored in the proof of Proposition 4.1, there is $w \in H_0^1(\Omega_{\Upsilon}) \setminus \{0\} \subset H^1(\Omega'_{\Upsilon})$ such that

$$w_j = w|_{\Omega_j} \neq 0 \quad \forall j \in \Upsilon$$

and

$$w_{n_k} \rightarrow w \text{ in } H^1(\Omega'_{\Upsilon}), \text{ as } k \rightarrow \infty.$$

Furthermore, we also can prove that

$$c_{\lambda_{n_k}, \Upsilon} = \phi_{\lambda_{n_k}, \Upsilon}(w_{n_k}) \rightarrow I_{\Upsilon}(w)$$

and

$$0 = \phi'_{\lambda_{n_k}, \Upsilon}(w_{n_k}) \rightarrow I'_{\Upsilon}(w).$$

Then, $w \in \mathcal{M}_{\Upsilon}$, and by definition of c_{Υ} ,

$$\lim_k c_{\lambda_{n_k}, \Upsilon} \geq c_{\Upsilon}.$$

The last inequality together with item (i) implies

$$c_{\lambda_{n_k}, \Upsilon} \rightarrow c_{\Upsilon}, \text{ as } k \rightarrow \infty.$$

This establishes the asserted result.

□

In the sequel, we fix $R > 1$ verifying

$$0 < I'_j \left(\frac{1}{R} w_j + \sum_{k=1, k \neq j}^l t_k R w_k \right) \left(\frac{1}{R} w_j \right) \text{ and } I'_\Upsilon \left(R w_j + \sum_{k=1, k \neq j}^l t_k R w_k \right) (R w_j) < 0, \quad (6.5)$$

for $j \in \Upsilon$ and $\forall t_k \in [1/R^2, 1]$ with $k \neq j$.

In the sequel, to simplify the notation, we rename the components Ω_j of Ω in way such that $\Upsilon = \{1, 2, \dots, l\}$ for some $1 \leq l \leq k$. Then, we define:

$$\gamma_0(\mathbf{t})(x) = \sum_{j=1}^l t_j R w_j(x) \in H_0^1(\Omega_\Upsilon), \forall \mathbf{t} = (t_1, \dots, t_l) \in [1/R^2, 1]^l, \\ \Gamma_* = \left\{ \gamma \in C([1/R^2, 1]^l, E_\lambda \setminus \{0\}); \gamma(\mathbf{t})|_{\Omega'_j} \neq 0 \ \forall j \in \Upsilon; \gamma = \gamma_0 \text{ on } \partial[1/R^2, 1]^l \right\}$$

and

$$b_{\lambda, \Upsilon} = \inf_{\gamma \in \Gamma_*} \max_{\mathbf{t} \in [1/R^2, 1]^l} \phi_\lambda(\gamma(\mathbf{t})).$$

Next, our intention is proving an important relation among $b_{\lambda, \Upsilon}$, c_Υ and $c_{\lambda, \Upsilon}$. However, to do this, we need to some technical lemmas. The arguments used are the same found in [1], however for reader's convenience we will repeat their proofs

Lemma 6.2. *For all $\gamma \in \Gamma_*$, there exists $(s_1, \dots, s_l) \in [1/R^2, 1]^l$ such that*

$$\phi'_{\lambda, \Upsilon}(\gamma(s_1, \dots, s_l))(\tilde{\gamma}_j(s_1, \dots, s_l)) = 0, \forall j \in \Upsilon$$

where

$$\tilde{\gamma}_j(t_1, \dots, t_l)(x) = \begin{cases} \gamma(t_1, \dots, t_l)(x), & x \in \Omega'_j \\ 0, & x \in \Omega'_\Upsilon \setminus \Omega'_j \end{cases}$$

Proof Given $\gamma \in \Gamma_*$, consider $\hat{\gamma}: [1/R^2, 1]^l \rightarrow \mathbb{R}^l$ such that

$$\hat{\gamma}(\mathbf{t}) = \left(\phi'_{\lambda, \Upsilon}(\gamma(\mathbf{t}))\tilde{\gamma}_1(\mathbf{t}), \dots, \phi'_{\lambda, \Upsilon}(\gamma(\mathbf{t}))\tilde{\gamma}_l(\mathbf{t}) \right), \text{ where } \mathbf{t} = (t_1, \dots, t_l).$$

For $\mathbf{t} \in \partial[1/R^2, 1]^l$, it holds

$$\hat{\gamma}(\mathbf{t}) = \hat{\gamma}_0(\mathbf{t}), \quad (6.6)$$

where

$$\hat{\gamma}_0(\mathbf{t}) = \left(I'_\Upsilon(\gamma_0(\mathbf{t}))t_1 R w_1, \dots, I'_\Upsilon(\gamma_0(\mathbf{t}))t_l R w_l \right).$$

Now, lemma follows using (6.5), (6.6) and Miranda's Theorem [30]. □

Proposition 6.3.

$$(i) \ c_{\lambda, \Upsilon} \leq b_{\lambda, \Upsilon} \leq c_\Upsilon, \forall \lambda \geq 1;$$

$$(ii) \ b_{\lambda, \Upsilon} \rightarrow c_\Upsilon, \text{ as } \lambda \rightarrow \infty;$$

(iii) $\phi_\lambda(\gamma(\mathbf{t})) < c_\Upsilon$, $\forall \lambda \geq 1, \gamma \in \Gamma_*$ and $\mathbf{t} = (t_1, \dots, t_l) \in \partial[1/R^2, 1]^l$.

Proof

(i) Since $\gamma_0 \in \Gamma_*$, by (6.3),

$$b_{\lambda, \Upsilon} \leq \max_{(t_1, \dots, t_l) \in [1/R^2, 1]^l} \phi_\lambda(\gamma_0(t_1, \dots, t_l)) \leq \max_{(t_1, \dots, t_l) \in [1/R^2, 1]^l} I_\Upsilon\left(\sum_{j=1}^l t_j R w_j\right) = c_\Upsilon.$$

Now, fixing $\mathbf{s} = (s_1, \dots, s_l) \in [1/R^2, 1]^l$ given in Lemma 6.2 and recalling that

$$c_{\lambda, \Upsilon} = \inf_{u \in \mathcal{M}'_\Upsilon} \phi_{\lambda, \Upsilon}(u)$$

where

$$\mathcal{M}'_\Upsilon = \{u \in \mathcal{N}'_\Upsilon : \phi'_{\lambda, \Upsilon}(u)u_j = 0 \text{ and } u_j \neq 0 \ \forall j \in \Upsilon\},$$

it follows that

$$\phi_{\lambda, \Upsilon}(\gamma(\mathbf{s})) \geq c_{\lambda, \Upsilon}.$$

From (3.8),

$$\phi_{\lambda, \mathbb{R}^3 \setminus \Omega'_\Upsilon}(u) \geq 0, \ \forall u \in H^1(\mathbb{R}^3 \setminus \Omega'_\Upsilon),$$

which leads to

$$\phi_\lambda(\gamma(\mathbf{t})) \geq \phi_{\lambda, \Upsilon}(\gamma(\mathbf{t})), \ \forall \mathbf{t} = (t_1, \dots, t_l) \in [1/R^2, 1]^l.$$

Thus

$$\max_{(t_1, \dots, t_l) \in [1/R^2, 1]^l} \phi_\lambda(\gamma(t_1, \dots, t_l)) \geq \phi_{\lambda, \Upsilon}(\gamma(\mathbf{s})) \geq c_{\lambda, \Upsilon},$$

showing that

$$b_{\lambda, \Upsilon} \geq c_{\lambda, \Upsilon}.$$

(ii) This limit is clear by the previous items, since we already know $c_{\lambda, \Upsilon} \rightarrow c_\Upsilon$, as $\lambda \rightarrow \infty$;

(iii) For $\mathbf{t} = (t_1, \dots, t_l) \in \partial[1/R^2, 1]^l$, it holds $\gamma(\mathbf{t}) = \gamma_0(\mathbf{t})$. From this,

$$\phi_\lambda(\gamma(\mathbf{t})) = I_\Upsilon(\gamma_0(\mathbf{t})).$$

From (6.4) and (6.5),

$$\phi_\lambda(\gamma(\mathbf{t})) \leq c_\Upsilon - \epsilon,$$

for some $\epsilon > 0$, so (iii) holds.

□

7 Proof of the main theorem

To prove Theorem 1.1, we need to find nonnegative solutions u_λ for large values of λ , which converges to a least energy solution of $(P)_{\infty, \Upsilon}$ as $\lambda \rightarrow \infty$. To this end, we will show two propositions which together with the Propositions 4.1 and 5.1 will imply that Theorem 1.1 holds.

Henceforth, we denote by

$$\Theta = \left\{ u \in E_\lambda : \|u\|_{\lambda, \Omega'_j} > \frac{\tau}{8R} \quad \forall j \in \Upsilon \right\}$$

and

$$\phi_\lambda^{c_\Upsilon} = \{ u \in E_\lambda : \phi_\lambda(u) \leq c_\Upsilon \}.$$

Moreover, we fix $\delta = \frac{\tau}{48R}$, $\mu > 0$ and

$$\mathcal{A}_\mu^\lambda = \{ u \in \Theta_{2\delta} : |\phi_\lambda(u) - c_\Upsilon| \leq \mu \}. \quad (7.1)$$

We observe that

$$w_\Upsilon \in \mathcal{A}_\mu^\lambda \cap \phi_\lambda^{c_\Upsilon},$$

showing that $\mathcal{A}_\mu^\lambda \cap \phi_\lambda^{c_\Upsilon} \neq \emptyset$.

Our next result shows an important uniform estimate of $\|\phi'_\lambda(u)\|_{E_\lambda^*}$ on the region $(\mathcal{A}_{2\mu}^\lambda \setminus \mathcal{A}_\mu^\lambda) \cap \phi_\lambda^{c_\Upsilon}$.

Proposition 7.1. *For each $\mu > 0$, there exist $\Lambda_* \geq 1$ and $\sigma_0 > 0$ independent of λ such that*

$$\|\phi'_\lambda(u)\|_{E_\lambda^*} \geq \sigma_0, \text{ for } \lambda \geq \Lambda_* \text{ and all } u \in (\mathcal{A}_{2\mu}^\lambda \setminus \mathcal{A}_\mu^\lambda) \cap \phi_\lambda^{c_\Upsilon}. \quad (7.2)$$

Proof We assume that there exist $\lambda_n \rightarrow \infty$ and $u_n \in (\mathcal{A}_{2\mu}^{\lambda_n} \setminus \mathcal{A}_\mu^{\lambda_n}) \cap \phi_{\lambda_n}^{c_\Upsilon}$ such that

$$\|\phi'_{\lambda_n}(u_n)\|_{E_{\lambda_n}^*} \rightarrow 0.$$

Since $u_n \in \mathcal{A}_{2\mu}^{\lambda_n}$, this implies $(\|u_n\|_{\lambda_n})$ is a bounded sequence and, consequently, it follows that $(\phi_{\lambda_n}(u_n))$ is also bounded. Thus, passing a subsequence if necessary, we can assume that $(\phi_{\lambda_n}(u_n))$ converges. Thus, from Proposition 4.1, there exists $0 \leq u \in H_0^1(\Omega_\Upsilon)$ such that u is a solution for $(P)_\Upsilon$,

$$u_n \rightarrow u \text{ in } H^1(\mathbb{R}^3), \quad \|u_n\|_{\lambda_n, \mathbb{R}^3 \setminus \Omega_\Upsilon} \rightarrow 0 \text{ and } \phi_{\lambda_n}(u_n) \rightarrow I_\Upsilon(u).$$

Recalling that $(u_n) \subset \Theta_{2\delta}$, we derive that

$$\|u_n\|_{\lambda_n, \Omega'_j} > \frac{\tau}{12R} \quad \forall j \in \Upsilon.$$

Then, taking the limit of $n \rightarrow +\infty$, we find

$$\|u\|_j \geq \frac{\tau}{12R} \quad \forall j \in \Upsilon,$$

yields $u|_{\Omega_j} \neq 0$ for all $j \in \Upsilon$ and $I'_\Upsilon(u) = 0$. Consequently, by (6.1),

$$\|u\|_j > \frac{\tau}{8R} \quad \forall j \in \Upsilon.$$

This way, $I_\Upsilon(u) \geq c_\Upsilon$. But since $\phi_{\lambda_n}(u_n) \leq c_\Upsilon$ and $\phi_{\lambda_n}(u_n) \rightarrow I_\Upsilon(u)$, for n large, it holds

$$\|u_n\|_j > \frac{\tau}{8R} \quad \text{and} \quad |\phi_{\lambda_n}(u_n) - c_\Upsilon| \leq \mu, \quad \forall j \in \Upsilon.$$

So $u_n \in \mathcal{A}_\mu^{\lambda_n}$, obtaining a contradiction. Thus, we have completed the proof. \square

In the sequel, μ_1, μ^* denote the following numbers

$$\min_{\mathbf{t} \in \partial[1/R^2, 1]^l} |I_\Upsilon(\gamma_0(\mathbf{t})) - c_\Upsilon| = \mu_1 > 0$$

and

$$\mu^* = \min\{\mu_1, \delta, r/2\},$$

where δ were given (7.1) and

$$r = R^2 \left(\frac{1}{2} - \frac{1}{\theta} \right)^{-1} c_\Upsilon.$$

Moreover, for each $s > 0$, B_s^λ denotes the set

$$B_s^\lambda = \{u \in E_\lambda; \|u\|_\lambda \leq s\} \quad \text{for } s > 0.$$

Proposition 7.2. *Let $\mu > 0$ small enough and $\Lambda_* \geq 1$ given in the previous proposition. Then, for $\lambda \geq \Lambda_*$, there exists a solution u_λ of (A_λ) such that $u_\lambda \in \mathcal{A}_\mu^\lambda \cap \phi_\lambda^{c_\Upsilon} \cap B_{r+1}^\lambda$.*

Proof Let $\lambda \geq \Lambda_*$. Assume that there are no critical points of ϕ_λ in $\mathcal{A}_\mu^\lambda \cap \phi_\lambda^{c_\Upsilon} \cap B_{r+1}^\lambda$. Since ϕ_λ verifies the (PS) condition, there exists a constant $d_\lambda > 0$ such that

$$\|\phi'_\lambda(u)\|_{E_\lambda^*} \geq d_\lambda, \quad \text{for all } u \in \mathcal{A}_\mu^\lambda \cap \phi_\lambda^{c_\Upsilon} \cap B_{r+1}^\lambda.$$

From Proposition 7.1,

$$\|\phi'_\lambda(u)\|_{E_\lambda^*} \geq \sigma_0, \quad \text{for all } u \in \left(\mathcal{A}_{2\mu}^\lambda \setminus \mathcal{A}_\mu^\lambda \right) \cap \phi_\lambda^{c_\Upsilon},$$

where $\sigma_0 > 0$ does not depend on λ . In what follows, $\Psi: E_\lambda \rightarrow \mathbb{R}$ is a continuous functional verifying

$$\Psi(u) = 1, \quad \text{for } u \in \mathcal{A}_{\frac{3}{2}\mu}^\lambda \cap \Theta_\delta \cap B_r^\lambda,$$

$$\Psi(u) = 0, \quad \text{for } u \notin \mathcal{A}_{2\mu}^\lambda \cap \Theta_{2\delta} \cap B_{r+1}^\lambda$$

and

$$0 \leq \Psi(u) \leq 1, \quad \forall u \in E_\lambda.$$

We also consider $H: \phi_\lambda^{c_\Upsilon} \rightarrow E_\lambda$ given by

$$H(u) = \begin{cases} -\Psi(u) \|Y(u)\|^{-1} Y(u), & \text{for } u \in \mathcal{A}_{2\mu}^\lambda \cap B_{r+1}^\lambda, \\ 0, & \text{for } u \notin \mathcal{A}_{2\mu}^\lambda \cap B_{r+1}^\lambda, \end{cases}$$

where Y is a pseudo-gradient vector field for Φ_λ on $\mathcal{K} = \{u \in E_\lambda; \phi'_\lambda(u) \neq 0\}$. Observe that H is well defined, once $\phi'_\lambda(u) \neq 0$, for $u \in \mathcal{A}_{2\mu}^\lambda \cap \phi_\lambda^{c_\Upsilon}$. The inequality

$$\|H(u)\| \leq 1, \forall \lambda \geq \Lambda_* \text{ and } u \in \phi_\lambda^{c_\Upsilon},$$

guarantees that the deformation flow $\eta: [0, \infty) \times \phi_\lambda^{c_\Upsilon} \rightarrow \phi_\lambda^{c_\Upsilon}$ defined by

$$\frac{d\eta}{dt} = H(\eta), \quad \eta(0, u) = u \in \phi_\lambda^{c_\Upsilon}$$

verifies

$$\frac{d}{dt}\phi_\lambda(\eta(t, u)) \leq -\frac{1}{2}\Psi(\eta(t, u))\|\phi'_\lambda(\eta(t, u))\| \leq 0, \quad (7.3)$$

$$\left\|\frac{d\eta}{dt}\right\|_\lambda = \|H(\eta)\|_\lambda \leq 1 \quad (7.4)$$

and

$$\eta(t, u) = u \text{ for all } t \geq 0 \text{ and } u \in \phi_\lambda^{c_\Upsilon} \setminus \mathcal{A}_{2\mu}^\lambda \cap B_{r+1}^\lambda. \quad (7.5)$$

Next, we study two paths, which are relevant for what follows:

- The path $\mathbf{t} \mapsto \eta(t, \gamma_0(\mathbf{t}))$, where $\mathbf{t} = (t_1, \dots, t_l) \in [1/R^2, 1]^l$.

Thereby, if $\mu \in (0, \mu^*)$, we have that

$$\gamma_0(\mathbf{t}) \notin \mathcal{A}_{2\mu}^\lambda, \forall \mathbf{t} \in \partial[1/R^2, 1]^l.$$

Since

$$\phi_\lambda(\gamma_0(\mathbf{t})) < c_\Upsilon, \forall \mathbf{t} \in \partial[1/R^2, 1]^l,$$

from (7.5), it follows that

$$\eta(t, \gamma_0(\mathbf{t})) = \gamma_0(\mathbf{t}), \forall \mathbf{t} \in \partial[1/R^2, 1]^l.$$

So, $\eta(t, \gamma_0(\mathbf{t})) \in \Gamma_*$, for each $t \geq 0$.

- The path $\mathbf{t} \mapsto \gamma_0(\mathbf{t})$, where $\mathbf{t} = (t_1, \dots, t_l) \in [1/R^2, 1]^l$.

We observe that

$$\text{supp}(\gamma_0(\mathbf{t})) \subset \overline{\Omega_\Upsilon}$$

and

$$\phi_\lambda(\gamma_0(\mathbf{t})) \text{ does not depend on } \lambda \geq 1,$$

for all $\mathbf{t} \in [1/R^2, 1]^l$. Moreover,

$$\phi_\lambda(\gamma_0(\mathbf{t})) \leq c_\Upsilon, \forall \mathbf{t} \in [1/R^2, 1]^l$$

and

$$\phi_\lambda(\gamma_0(\mathbf{t})) = c_\Upsilon \text{ if, and only if, } t_j = \frac{1}{R}, \forall j \in \Upsilon.$$

Therefore

$$m_0 = \sup \left\{ \phi_\lambda(u); u \in \gamma_0([1/R^2, 1]^l) \setminus A_\mu^\lambda \right\}$$

is independent of λ and $m_0 < c_\Upsilon$. Now, observing that there exists $K_* > 0$ such that

$$|\phi_\lambda(u) - \phi_\lambda(v)| \leq K_* \|u - v\|_\lambda, \forall u, v \in \mathcal{B}_r^\lambda,$$

we derive

$$\max_{\mathbf{t} \in [1/R^2, 1]^l} \phi_\lambda(\eta(T, \gamma_0(\mathbf{t}))) \leq \max \left\{ m_0, c_\Upsilon - \frac{1}{2K_*} \sigma_0 \mu \right\}, \quad (7.6)$$

for $T > 0$ large.

In fact, writing $u = \gamma_0(\mathbf{t})$, $\mathbf{t} \in [1/R^2, 1]^l$, if $u \notin A_\mu^\lambda$, from (7.3),

$$\phi_\lambda(\eta(t, u)) \leq \phi_\lambda(u) \leq m_0, \forall t \geq 0,$$

and we have nothing more to do. We assume then $u \in A_\mu^\lambda$ and set

$$\tilde{\eta}(t) = \eta(t, u), \quad \widetilde{d_\lambda} = \min \{d_\lambda, \sigma_0\} \quad \text{and} \quad T = \frac{\sigma_0 \mu}{K_* \widetilde{d_\lambda}}.$$

Now, we will analyze the ensuing cases:

Case 1: $\tilde{\eta}(t) \in \mathcal{A}_{\frac{3}{2}\mu}^\lambda \cap \Theta_\delta \cap B_r^\lambda, \forall t \in [0, T]$.

Case 2: $\tilde{\eta}(t_0) \notin \mathcal{A}_{\frac{3}{2}\mu}^\lambda \cap \Theta_\delta \cap B_r^\lambda$, for some $t_0 \in [0, T]$.

Analysis of Case 1

In this case, we have $\Psi(\tilde{\eta}(t)) = 1$ and $\|\phi'_\lambda(\tilde{\eta}(t))\| \geq \widetilde{d_\lambda}$ for all $t \in [0, T]$. Hence, from (7.3),

$$\phi_\lambda(\tilde{\eta}(T)) = \phi_\lambda(u) + \int_0^T \frac{d}{ds} \phi_\lambda(\tilde{\eta}(s)) ds \leq c_\Upsilon - \frac{1}{2} \int_0^T \widetilde{d_\lambda} ds,$$

that is,

$$\phi_\lambda(\tilde{\eta}(T)) \leq c_\Upsilon - \frac{1}{2} \widetilde{d_\lambda} T = c_\Upsilon - \frac{1}{2K_*} \sigma_0 \mu,$$

showing (7.6).

Analysis of Case 2: In this case we have the following situations:

(a): There exists $t_2 \in [0, T]$ such that $\tilde{\eta}(t_2) \notin \Theta_\delta$, and thus, for $t_1 = 0$ it follows that

$$\|\tilde{\eta}(t_2) - \tilde{\eta}(t_1)\| \geq \delta > \mu,$$

because $\tilde{\eta}(t_1) = u \in \Theta$.

(b): There exists $t_2 \in [0, T]$ such that $\tilde{\eta}(t_2) \notin B_r^\lambda$, so that for $t_1 = 0$, we get

$$\|\tilde{\eta}(t_2) - \tilde{\eta}(t_1)\| \geq r > \mu,$$

because $\tilde{\eta}(t_1) = u \in B_r^\lambda$.

(c): $\tilde{\eta}(t) \in \Theta_\delta \cap B_r^\lambda$ for all $t \in [0, T]$, and there are $0 \leq t_1 \leq t_2 \leq T$ such that $\tilde{\eta}(t) \in \mathcal{A}_{\frac{3}{2}\mu}^\lambda \setminus \mathcal{A}_\mu^\lambda$ for all $t \in [t_1, t_2]$ with

$$|\phi_\lambda(\tilde{\eta}(t_1)) - c_\Upsilon| = \mu \text{ and } |\phi_\lambda(\tilde{\eta}(t_2)) - c_\Upsilon| = \frac{3\mu}{2}$$

From definition of K_* , we have

$$\|w_2 - w_1\| \geq \frac{1}{K_*} |\phi_\lambda(w_2) - \phi_\lambda(w_1)| \geq \frac{1}{2K_*} \mu.$$

Then, by mean value theorem, $t_2 - t_1 \geq \frac{1}{2K_*} \mu$ and, this form,

$$\phi_\lambda(\tilde{\eta}(T)) \leq \phi_\lambda(u) - \int_0^T \Psi(\tilde{\eta}(s)) \|\phi'_\lambda(\tilde{\eta}(s))\| ds$$

implying

$$\phi_\lambda(\tilde{\eta}(T)) \leq c_\Upsilon - \int_{t_1}^{t_2} \sigma_0 ds = c_\Upsilon - \sigma_0(t_2 - t_1) \leq c_\Upsilon - \frac{1}{2K_*} \sigma_0 \mu,$$

which proves (7.6). Fixing $\hat{\eta}(t_1, \dots, t_l) = \eta(T, \gamma_0(t_1, \dots, t_l))$, we have that $\hat{\eta}(t_1, \dots, t_l) \in \Theta_{2\delta}$, and so, $\hat{\eta}(t_1, \dots, t_l)|_{\Omega'_j} \neq 0$ for all $j \in \Upsilon$. Thus, $\hat{\eta} \in \Gamma_*$, leading to

$$b_{\lambda, \Gamma} \leq \max_{(t_1, \dots, t_l) \in [1/R^2, 1]} \phi_\lambda(\hat{\eta}(t_1, \dots, t_l)) \leq \max \left\{ m_0, c_\Upsilon - \frac{1}{2K_*} \sigma_0 \mu \right\} < c_\Upsilon,$$

which contradicts the fact that $b_{\lambda, \Gamma} \rightarrow c_\Upsilon$. \square

[Proof of Theorem 1.1] According Proposition 7.2, for $\mu \in (0, \mu^*)$ and $\Lambda_* \geq 1$, there exists a solution u_λ for (A_λ) such that $u_\lambda \in \mathcal{A}_\mu^\lambda \cap \phi_\lambda^{c_\Upsilon}$, for all $\lambda \geq \Lambda_*$.

Claim: There are $\lambda_0 \geq \Lambda_*$ and $\mu_0 > 0$ small enough, such that u_λ is a solution for $(P)_\lambda$ for $\lambda \geq \lambda_0$ and $\mu \in (0, \mu_0)$.

Indeed, fixed $\mu \in (0, \mu_0)$, assume by contradiction that there are $\lambda_n \rightarrow \infty$, such that (u_{λ_n}) is not a solution for $(P)_{\lambda_n}$. From Proposition 7.2, the sequence (u_{λ_n}) verifies:

- (a) $\phi'_{\lambda_n}(u_{\lambda_n}) = 0, \forall n \in \mathbb{N}$;
- (b) $\|u_n\|_{\lambda_n, \mathbb{R}^3 \setminus \Omega_\Upsilon}^2(u_{\lambda_n}) \rightarrow 0$;
- (c) $\phi_{\lambda_n}(u_{\lambda_n}) \rightarrow d \leq c_\Upsilon$.

The item (b) ensures we can use Proposition 5.1 to deduce u_{λ_n} is a solution for $(P)_{\lambda_n}$, for large values of n , which is a contradiction, showing this way the claim.

Now, our goal is to prove the second part of the theorem. To this end, let (u_{λ_n}) be a sequence verifying the above limits. A direct computation gives $\phi_{\lambda_n}(u_{\lambda_n}) \rightarrow d$ with

$d \leq c_\Upsilon$. This way, using Proposition 4.1 combined with item (c), we derive (u_{λ_n}) converges in $H^1(\mathbb{R}^3)$ to a function $u \in H^1(\mathbb{R}^3)$, which satisfies $u = 0$ outside Ω_Υ and $u|_{\Omega_j} \neq 0$, $j \in \Upsilon$, and u is a positive solution for

$$\begin{cases} M\left(\int_{\Omega_\Upsilon} (|\nabla u|^2 + u^2)dx\right)(-\Delta u + u) = f(u), & \text{in } \Omega_\Upsilon, \\ u \in H_0^1(\Omega_\Upsilon), \end{cases} \quad (P)_{\infty, \Upsilon}$$

and so,

$$I_\Upsilon(u) \geq c_\Upsilon.$$

On the other hand, we also know that

$$\phi_{\lambda_n}(u_{\lambda_n}) \rightarrow I_\Upsilon(u),$$

implying that

$$I_\Upsilon(u) = d \text{ and } d \geq c_\Upsilon.$$

Since $d \leq c_\Upsilon$, we deduce that

$$I_\Upsilon(u) = c_\Upsilon,$$

showing that u is a least energy solution for $(P)_{\infty, \Upsilon}$. Consequently, u is a least energy solution for the problem

$$\begin{cases} M\left(\int_{\Omega_\Upsilon} (|\nabla u|^2 + u^2)dx\right)(-\Delta u + u) = f(u), & \text{in } \Omega_\Upsilon, \\ u \in H_0^1(\Omega_\Upsilon). \end{cases}$$

□

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